IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

On ladder representations and affinisation of Lie algebras

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1988 J. Phys. A: Math. Gen. 21 885

(http://iopscience.iop.org/0305-4470/21/4/017)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 31/05/2010 at 15:36

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **21** (1988) 885-893. Printed in the UK

On ladder representations and affinisation of Lie algebras

D T Stoyanov
Institute of Nuclear Research and Nuclear Energy, Boulevard Lenin 72, Sofia, Bulgaria

Received 30 July 1987

Abstract. A generalisation of the procedure for constructing ladder representations of finite and infinite-dimensional Lie algebras is considered. Such a general procedure is further generalised and provides a method for affinisation of Lie algebras. A general form of the central term for arbitrary Lie algebra is obtained.

Ladder representations have found different applications in physics as some of the simplest representations having the most clear meaning from the physical point of view. Usually these were particular applications for some concrete algebras. Some of the papers considering questions connected with the ladder representations are cited in Stoyanov and Todorov (1968) in which some examples are also considered. In the present paper we try to give a somewhat more general definition of the ladder representations which must be suitable for a future generalisation. Such generalisation is given at the end and applied for some Lie algebras.

We will start with the definition of ladder representations of a (finite or infinite-dimensional) Lie algebra. In general these are the realisation of the algebra generators by suitable creation and annihilation operators acting on the corresponding Fock space. In this sense the space on which the algebra operators in a ladder representation act is in general a subspace of that Fock space.

Let us consider the construction of these representations in more detail. With J_A we denote the generators of the initial real Lie algebra which satisfy the following commutation relations:

$$[J_A, J_B] = f^C_{AB} J_C \tag{1}$$

where f^C_{AB} are the structure constants. The indices A, B, C may take finite or infinite values depending on the dimension of the Lie algebra.

Suppose that a matrix representation I_A of the operators J_A is given and the corresponding matrix elements are denoted by $(I_A)_{ik}$. The latin letters can take suitable integer of half-integer values. The matrices I_A can be finite or infinite dimensional. The next assumption is that the module M of this representation has an invariant non-generated quadratic form with a matric tensor denoted by G, with corresponding matrix elements G_{ik}. From the invariance we may write down the following relation:

$$I^+_A G + G I_A = 0 \tag{2}$$

for every I_A. I^+_A denotes the Hermitian conjugated matrix.

Now let us introduce the set of creation and annihilation operators as operator-valued components of the vectors from M. In particular, by φ_i we denote the annihilation and with φ^*_i the corresponding creation operators. The commutation relations are as follows:

$$[\varphi, \varphi^*] = G^{-1}. \tag{3}$$

The asterisk denotes Hermitian conjugation of the operators φ.

The Fock space appears, as usual, by introducing the vacuum state $|0\rangle$ with the help of which one can construct the one-particle states $\varphi_i^*|0\rangle$, the two-particle states $\varphi_i^*\varphi_\kappa^*|0\rangle$, etc.

Proposition 1. The operators

$$J_A = \varphi^* G I_A \varphi \equiv \varphi_i^* G_{i\kappa} (I_A)_{\kappa l} \varphi_l \tag{4}$$

acting on the Fock space F form a representation of the algebra (1).

With the help of equation (3) it is easy to prove this proposition by direct calculation. Indeed, we can obtain in this way the following identity:

$$[J_A, J_B] = \varphi^* G [I_A, I_B] \varphi. \tag{5}$$

Because the matrices I_A satisfy the relations (1), from (5) we obtain that the operators J_A satisfy them too.

In general, this representation is reducible. From the identity (5) we can see that the operator

$$C_1 = \varphi^* G \varphi \tag{6}$$

commutes with all generators J_A. This is the first-order Casimir operator which has the meaning of particle number operator. Then we see that the Fock space F reduces to an infinite set of subspaces on every one of which an irreducible representation of our algebra acts. Every such subspace is characterised by a definite eigenvalue of the Casimir operator (6). In particular, the vacuum state realises the scalar representation of the algebra and the eigenvalue of the Casimir operator in this case is zero. The one-particle states realise another representation for which the eigenvalue of the Casimir operator is one. The same is valid for the two-particle states, three-particle states, etc.

Because of this special structure of the representation in the Fock space F, they were named ladder representations. In our paper these will be called ladder representations of the usual type.

The ladder representations of the usual type have been considered in a large number of papers in connection with different physical applications. For the algebras of the groups SU(2, 2) and SU(6, 6), for example, one may see the papers of Stoyanov and Todorov (1967, 1968) and Todorov (1966).

We may obtain a different form of the ladder representation if we assume that there is a connection between the operators φ_i and their conjugates φ_i^*,

$$\varphi_i^* = \varphi_\kappa V_{\kappa i} \tag{7}$$

where V is an invertible matrix. In this case we cannot consider φ_i and φ_i^* anymore as annihilation and creation operators. Moreover, the matrix V exists if and only if the matrix representation I_A is equivalent to a real representation. Indeed we may rewrite the relation (7) for the transformed operators φ_i' and $\varphi_i^{*\prime}$:

$$\varphi_i^{*\prime} = \varphi_\kappa' V_{\kappa i} \tag{8}$$

where

$$\varphi' = I_A \varphi \qquad \varphi^{*\prime} = \varphi^* I_A^+. \tag{9}$$

Then from (7)-(9) we obtain

$$VI_A^+V^{-1}=I_A^{\mathrm{T}} \tag{10}$$

and

$$V^{-1}=\bar{V} \tag{11}$$

where the superscript T stands for transposition and $\bar{V}$ denotes the complex conjugate matrix.

In this case we have new commutation relations between the components of the operator φ which can be obtained from (3) and (7):

$$[\varphi_i, \varphi_\kappa]=(G^{-1}\bar{V})_{i\kappa}=(G^{-1}V^{-1})_{i\kappa}. \tag{12}$$

The structure of these relations shows that the annihilation and creation operators appear now as different components of the same vector φ. One may construct the simplest example of such an operator φ by just joining together the sets of annihilation and creation operators.

Then we can prove the following.

Proposition 2. The operators:

$$L_A=\tfrac{1}{2}\varphi VGI_A\varphi \tag{13}$$

form a representation of the algebra (1).

To verify this proposition we must also make some calculations which are more complicated in this case. Indeed with direct calculations we obtain for the right-hand side of the relations (1):

$$[L_A, L_B]=\tfrac{1}{4}\varphi VG[I_A, I_B]\varphi+\tfrac{1}{4}(I_B\varphi)_i(VGI_A\varphi)_i-\tfrac{1}{4}(\varphi VGI_A)_i(\varphi VGI_BG^{-1}V^{-1})_i. \tag{14}$$

Then with the help of (2) and (10) the equality (14) can be written down in the following form:

$$[L_A, L_B]=\tfrac{1}{2}\varphi VG[I_A, I_B]\varphi.$$

From the last identity the validity of the proposition (2) becomes obvious.

One of the main properties of this type of ladder representation is that the first-order Casimir operator

$$C_1=\varphi VG\varphi \tag{15}$$

has a fixed value.

From the commutation relations (12) we can find this value. Multiplying this relation with the matrix VG and taking traces we obtain that

$$C_1=\tfrac{1}{2}\operatorname{Tr}1=\tfrac{1}{2}d \tag{16}$$

where d is the dimension of the matrix representation, for example, the dimension of the space M. To find the result (16) we have taken into account that the matrix VG is antisymmetric because of (12).

It seems to us that the ladder representations of this second type have been considered only rarely and have not been completely investigated. In Stoyanov and Todorov (1968), in particular, representations of this type have been applied for the Lie algebra of the group Sp(4, R).

Next we shall generalise the ladder representation method described above. With the help of this generalisation the ladder representations may be used not only for the construction of new representations of a given Lie algebra but also for the representations of the corresponding Kac–Moody algebra.

We begin from the same assumption as above about the matrix representation of the Lie algebra (1). But here we suppose that there are infinite number of modules M_α of the same type of matrix representation, where the Greek letters may take every integer values (i.e. $\alpha \in \mathbb{Z}$). The representation matrices I_A from different modules M_α coincide with each other. We suppose that M_α is equivalent to the real module, but after conjugation its vectors can pass to another M_β with the help of a splitting matrix $V_{ik}k_{\alpha\beta}$. Then the annihilation and creation operators are the operator-valued components of the infinite-dimensional vector from the direct sum of M. Let us denote this vector by φ with components $\varphi_{i\alpha}$. Here a latin index i indicates the number of vector components in one module whose number is α.

Remark. In all modules, M_α bases of the same type have been chosen. This means that all matrices such as G in the different M_α also coincide with each other.

Now we can define the commutation relation between arbitrary components of the infinite-dimensional vector operator φ:

$$[\varphi_{i\alpha}, \varphi_{j\beta}] = (G^{-1}V^{-1})_{ij}k_{\alpha\beta}. \tag{17}$$

The Fock space may be constructed in the usual way by defining the vacuum state $|0\rangle$ and extracting the pairs of creation and annihilation operators from the vector φ. If this is done we can define the normal product of the operators $\varphi_{i\alpha}$:

$$:\varphi_{i\alpha}\varphi_{j\beta}:$$

in which every annihilation operator stands to the right of every creation operator. Let us denote the difference between the usual and normal products as follows:

$$D_{i\alpha;j\beta} = \varphi_{i\alpha}\varphi_{j\beta} - :\varphi_{i\alpha}\varphi_{j\beta}:. \tag{18}$$

In every concrete case the matrix $D_{i\alpha;j\beta}$ can be defined. In general it must satisfy the following identity:

$$D_{i\alpha;j\beta} - D_{j\beta;i\alpha} = (G^{-1}V^{-1})_{ij}k_{\alpha\beta}. \tag{19}$$

Each product of the operators $\varphi_{i\alpha}$ can be brought into the normal form by using the Wick theorem.

Theorem. Let us define the operators

$$\tilde{L}_{A,\xi} = \tfrac{1}{2}:\varphi_{i\alpha}(VGI_A)_{i\kappa}h^{\xi}_{\alpha\beta}\varphi_{\kappa\beta}: \tag{20}$$

where $(VG)_{ij}$ is the matrix, the inverse of which stands in the right-hand side of (17), I_A is the representation matrix in each of the modules M_α, and the matrices $h^{\xi}_{\alpha\beta}$ (ξ indicates the number of each matrix) are symmetric and satisfy the following relations:

$$h^{\xi}kh^{\eta} = h^{\xi+\eta}. \tag{21}$$

Then the operators $\tilde{L}_{A,\xi}$ from (20) satisfy the following commutation relations:

$$[\tilde{L}_{A,\xi}, \tilde{L}_{B,\eta}] = f^{C}_{AB}\tilde{L}_{C,\xi+\eta} + Q_{\xi,\eta}(I_A, I_B). \tag{22}$$

Here f_{AB}^{C} are the same structure constants as in (1); $Q_{\xi,\eta}(I_A, I_B)$ are c-number quantities representing the central charges and having the following form:

$$\begin{aligned} Q_{\xi,\eta}(I_A, I_B) = {} & \tfrac{1}{4} D_{i\alpha;\kappa\delta}(VGI_A)_{ij}(VGI_B)_{\kappa l} h^{\xi}_{\alpha\beta} h^{\eta}_{\delta\varepsilon} D_{j\beta;l\varepsilon} \\ & + \tfrac{1}{4} D_{i\alpha;l\varepsilon}(VGI_A)_{ij}(VGI_B)_{\kappa l} h^{\xi}_{\alpha\beta} h^{\eta}_{\delta\varepsilon} D_{j\beta;\kappa\delta} \\ & - \tfrac{1}{4} D_{\kappa\delta;i\alpha}(VGI_A)_{ij}(VGI_B)_{\kappa l} h^{\xi}_{\alpha\beta} h^{\eta}_{\delta\varepsilon} D_{l\varepsilon;j\beta} \\ & - \tfrac{1}{4} D_{l\varepsilon;i\alpha}(VGI_A)_{ij}(VGI_B)_{\kappa l} h^{\xi}_{\alpha\beta} h^{\eta}_{\delta\varepsilon} D_{\kappa\delta;j\beta}. \end{aligned} \tag{23}$$

Let us make some comments on this theorem. First of all the commutation relations (22) in general do not coincide with the ones of the initial Lie algebra and they define a new infinite-dimensional algebra of the Kac-Moody type. Because the central charges $Q_{\xi,\eta}(I_A, I_B)$ in the relation (22) are in a general form we can call the algebras of this type generalised Kac-Moody algebras. We shall see that most of the known algebras such as the usual Kac-Moody and Virasoro algebras may be obtained from equation (22). In this sense our theorem shows how to do what is called the affinisation of the given Lie algebra. It is important to note that the latter may be both finite and infinite dimensional.

From the relations (21) we can see that there are four possibilities for the values of the numbers ξ of the matrices h^{ξ}:

$$\xi = 0 \qquad \xi > 0 \qquad \xi < 0 \qquad \xi \in \mathbb{Z}$$

e.g. ξ takes all integer values. As we shall see later, the first case is suitable when the initial algebra is an infinite one.

If we suppose that

$$h^{\xi} = k^{-1} u^{\xi} \tag{24}$$

then we obtain from (21) for the matrices u^{ξ} the following

$$u^{\xi} u^{\eta} = u^{\xi+\eta}. \tag{25}$$

It is easy to solve these equations. Indeed the general solution has the form

$$u^{\xi} = (g)^{\xi} \tag{26}$$

where g is an arbitrary matrix.

The proof of our theorem may be obtained by direct calculations as above. Some complications arise in this case because of the normal product in the definition (20). But these calculations do not contain any unexpected difficulties and so we shall not perform them in this paper.

The first-order Casimir operator here also has a fixed value. If we define this operator as

$$C_1 = :\varphi_{i\alpha}(VG)_{ij} k^{-1}_{\alpha\beta} \varphi_{j\beta}: \tag{27}$$

i.e. with the help of a normal product, then its value is zero, because of the antisymmetry of the matrix $(VG)_{ij} k^{-1}_{\alpha\beta}$.

Now we consider some examples. First, we shall consider any finite-dimensional Lie algebra. In our opinion the most simple but non-trivial case appears if we take the Lie algebra of the group SL(2, R). Its generators have the form

$$(I_A)_{jk} = j\delta_{A+j,\kappa} \tag{28}$$

where $A = -1, 0, 1$ and $j, \kappa = \pm\frac{1}{2}$. They satisfy the following commutation relations:

$$[I_{\pm 1}, I_0] = \pm I_{\pm 1} \qquad [I_1, I_{-1}] = \tfrac{1}{2} I_0. \tag{29}$$

The metric tensor has the form:

$$G_{j,\kappa} = \frac{1}{j}\,\delta_{j+\kappa,0}. \tag{30}$$

The commutation relations (17) can be rewritten here in the following way:

$$[\varphi_{j\alpha}, \varphi_{\kappa\beta}] = -j\delta_{j+\kappa,0}\delta_{\alpha+\beta,0} \tag{31}$$

where α, β take all integer values. These commutation relations show that we have chosen the following type of the matrices:

$$\begin{aligned}(G^{-1}V^{-1})_{j\kappa} &= -j\delta_{j+\kappa,0}\\ k_{\alpha\beta} &= \delta_{\alpha+\beta,0}.\end{aligned} \tag{32}$$

Moreover, for the conjugate operators we have

$$\varphi^*_{i,d} = \varphi_{i,-\alpha}. \tag{33}$$

This relation shows how to define the pair of creation and annihilation operators. Without going in detail we write down only the matrix $D_{j\alpha;\kappa\beta}$ in this case:

$$D_{j\alpha;\kappa\beta} = -j\theta(\alpha)\delta_{j+\kappa,0}\delta_{\alpha+\beta,0} \tag{34}$$

where

$$\theta(\alpha) = \begin{cases} 1 & \alpha > 0\\ \frac{1}{2} & \alpha = 0\\ 0 & \alpha < 0\end{cases}$$

is the Heaviside function (with a defined value in $\alpha = 0$).

For the matrices h^ξ we choose here the following particular solution of (21):

$$h^\xi_{\alpha\beta} = \delta_{\xi,\alpha+\beta}$$

where ξ takes the same values as α and β.

Then we may write down the generators of the corresponding affinisation of the SL(2, R) Lie algebra. They have the form

$$\tilde{L}_{A\xi} = -\tfrac{1}{2}\sum_{\alpha=-\infty}^{\infty}\sum_{\kappa=\pm\frac{1}{2}} :\varphi_{\kappa,\alpha}\varphi_{A-\kappa,\xi-\alpha}:. \tag{35}$$

Of course, now we may verify that the operators (35) satisfy the commutation relations of the type (22), but this is not necessary. The only thing we must do is to calculate the central charge. In our case the matrices $(VGI_A)_{i\kappa}$ have the following form:

$$(VGI_A)_{\iota\kappa} = -\delta_{A-i,\kappa}. \tag{36}$$

We insert this and (34) into (23) to obtain:

$$\begin{aligned}Q_{\xi,\eta}(I_A, I_B) = \tfrac{1}{4}\sum_{\alpha,\delta=-\infty}^{\infty}\sum_{i,\kappa=\pm\frac{1}{2}} &(D_{i\alpha;\kappa\delta}D_{A-i,\xi-\alpha;B-\kappa,\eta-\delta}\\ &+ D_{i\alpha;B-\kappa,\eta-\delta}D_{A-i,\xi-\alpha;\kappa\delta} - D_{\kappa\delta,i\alpha}D_{B-\kappa,\eta-\delta;A-i,\xi-\alpha}\\ &- D_{B-\kappa,\eta-\delta;i\alpha}D_{\kappa\delta;A-i,\xi-\alpha}).\end{aligned} \tag{37}$$

If we now take into account that $D_{i\alpha;j\beta}$ is given by (34) then (37) can be rewritten as follows:

$$Q_{\xi,\eta}(I_A, I_B)=\tfrac{1}{2}\Bigg(\sum_{\alpha=-\infty}^{\infty}\sum_{\kappa=\pm\frac{1}{2}}[\kappa(A-\kappa)\theta(\alpha)\theta(\xi-\alpha)$$

$$-\kappa(A-\kappa)\theta(\alpha)\theta(-\xi-\alpha)]\Bigg)\delta_{\xi+\eta,0}\delta_{A+B,0}$$

$$=-\tfrac{1}{4}\xi\delta_{\xi+\eta,0}\delta_{A+B,0}. \tag{38}$$

One can see that after the affinisation we obtain the usual SL(2, R) Kac–Moody algebra with the well known central term.

The next example is connected with the infinite-dimensional Witt algebra. As is well known the generators of this algebra satisfy the following commutation relations:

$$[J_A, J_B]=(A-B)J_{A+B} \tag{39}$$

where A, B take all integer values. One can find very simple matrices which form a representation of this algebra. Here we have chosen the following ones:

$$(I_A)_{\kappa j}=\kappa\delta_{A+\kappa,j}. \tag{40}$$

The indices κ, j take all integer values except zero. To obtain the metric tensor one must first define the Hermitian conjugation of the matrix representation. As usual we suppose that

$$(I_A)^{+}=-I_{-A}^{T} \tag{41}$$

leading to the following metric tensor

$$G_{i\kappa}=\frac{1}{i}\,\delta_{i\kappa} \tag{42}$$

whereas the matrix V must have the form

$$V_{i\kappa}=\delta_{i+\kappa,0}. \tag{43}$$

Here we will not make any choice for the matrices $h^{\xi}_{\alpha\beta}$. This allows us to obtain a more general type of the central term. For the operators $\varphi_{i\alpha}$ we assume the following commutation relations:

$$[\varphi_{i\alpha}, \varphi_{j\beta}]=(G^{-1}V^{-1})_{ij}\delta_{\alpha\beta}\equiv -i\delta_{i+j,0}\delta_{\alpha\beta}. \tag{44}$$

As one can see, $k_{\alpha\beta}$ is chosen to be just $\delta_{\alpha\beta}$. Then let us write down the affine generators:

$$\tilde{L}_{A\xi}=\tfrac{1}{2}\sum_{\alpha,\beta=-\infty}^{\infty}\sum_{l\neq 0}:\varphi_{l\alpha}h^{\xi}_{\alpha\beta}\varphi_{A-l,\beta}:. \tag{45}$$

To calculate the central term we must define the pair of creation and annihilation operators or, which is the same, define the matrix $D_{i\alpha;j\beta}$. In our case the latter can be chosen as

$$D_{i\alpha;j\beta}=-i\theta(i)\delta_{i+j,0}\delta_{\alpha,\beta}. \tag{46}$$

This expression leads to a peculiar factorisation of the central term:

$$Q_{\xi,\eta}(I_A, I_B)=Q_{00}(I_A, I_B)\,\mathrm{Tr}\,h^{\xi+\eta} \tag{47}$$

where from (23) we have

$$Q_{00}(I_A, I_B) = \tfrac{1}{4}\Delta_{i\kappa}(VGI_A)_{ij}(VGI_B)_{\kappa l}\Delta_{jl} + \tfrac{1}{4}\Delta_{il}(VGI_A)_{ij}(VGI_B)_{\kappa l}\Delta_{j\kappa}$$
$$-\tfrac{1}{4}\Delta_{\kappa i}(VGI_A)_{ij}(VGI_B)_{\kappa l}\Delta_{lj} - \tfrac{1}{4}\Delta_{li}(VGI_A)_{ij}(VGI_B)_{\kappa l}\Delta_{\kappa j}. \tag{48}$$

Here we have denoted

$$\Delta_{i\kappa} = D_{i0;\kappa 0} \tag{49}$$

and in this case

$$\Delta_{i\kappa} = -i\theta(i)\delta_{i+\kappa,0}. \tag{50}$$

Using (40), (42), (43) and (50) we can obtain the specific value of $Q_{00}(I_A, I_B)$. Indeed,

$$Q_{00}(I_A, I_B) = \tfrac{1}{4}\sum_{i,\kappa\neq 0}\Delta_{i\kappa}\Delta_{A-i,B-\kappa} + \tfrac{1}{4}\sum_{i,\kappa\neq 0}\Delta_{i,B-\kappa}\Delta_{A-i,\kappa}$$
$$-\tfrac{1}{4}\sum_{i,\kappa\neq 0}\Delta_{\kappa,i}\Delta_{B-\kappa,A-i} - \tfrac{1}{4}\sum_{i,\kappa\neq 0}\Delta_{B-\kappa,i}\Delta_{\kappa,A-i}$$
$$= \tfrac{1}{2}\left(\sum_{i=-\infty}^{\infty}[i(A-i)\theta(i)\theta(A-i) + i(A+i)\theta(i)\theta(-A-i)]\right)\delta_{A+B,0}$$
$$= \tfrac{1}{12}A(A^2-1)\delta_{A+B,0}. \tag{51}$$

Then we obtain the final form of the central term:

$$Q_{\xi,\eta}(I_A, I_B) = \tfrac{1}{12}\,\mathrm{Tr}\,h^{\xi+\eta}\,A(A^2-1)\delta_{A+B,0}. \tag{52}$$

It is interesting to write down the new commutation relations of the generators $\tilde{L}_{a\xi}$ given by (45):

$$[\tilde{L}_{a\xi}, \tilde{L}_{b\xi}] = (A-B)L_{A+B,\xi+\eta} + \tfrac{1}{12}\,\mathrm{Tr}\,h^{\xi+\eta}\,A(A^2-1)\delta_{A+B,0}. \tag{53}$$

Algebras of this type were obtained for the first time by Stoyanov (1985) where we investigated the infinite Lie algebras connected with the four-dimensional Laplace equation. From (53) we can see several properties of the algebra obtained.

First, a particular case of equation (53) is the usual Virasoro algebra with an arbitrary value of the central charge. Indeed, if we had chosen the matrix g from (26) as a projection operator on any subspace of finite dimension d then we would have

$$h^{\xi} = (g)^{\xi} = g \tag{54}$$

for every ξ and

$$\mathrm{Tr}\,h^{\xi+\eta} = \mathrm{Tr}\,g = d.$$

In this case the generators $\tilde{L}_{a\xi}$ do not depend on ξ because of (54).

If the matrices $h^{\xi}_{\alpha\beta}$ are infinite dimensional but with finite traces, then the central charge is a function of the sum $\xi+\eta$. The exact form of this function depends on the choice of the matrix g. This case is very important because the construction of ladder representations of such algebras is the only thing we can do so far. Finally, let us note that such algebras must appear in the three-dimensional compactifications (maybe in any membrane theory) as the Virasoro algebra appears from the two-dimensional compactifications (e.g. in string theory).

Acknowledgments

It is a pleasure to thank I T Todorov, V Petkova, V Dobrev and L Hadjiivanov for illuminating discussions.

References

Stoyanov D T 1985 *Proc. Conf. on Conformal and Related Symmetries, Physical Results and Mathematical Background, Clausthal* (Berlin: Springer)
Stoyanov D T and Todorov I T 1967 *Ann. Phys., NY* **41** 349
—— 1968 *J. Math. Phys.* **9** 2146
Todorov I T 1966 *ICTP Trieste preprint* IC/66/71